

# On Independent Coalition in Graphs and Independent Coalition Graphs

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## Abstract

An independent coalition in a graph  $G$  consists of two disjoint, independent vertex sets  $V_1$  and  $V_2$ , such that neither  $V_1$  nor  $V_2$  is a dominating set, but the union  $V_1 \cup V_2$  is an independent dominating set of  $G$ . An independent coalition partition of  $G$  is a partition  $\{V_1, \dots, V_k\}$  of  $V(G)$  such that for every  $i \in [k]$ , either the set  $V_i$  consists of a single dominating vertex of  $G$ , or  $V_i$  forms an independent coalition with some other part  $V_j$ . The independent coalition number  $IC(G)$  of  $G$  is the maximum order of an independent coalition of  $G$ . The independent coalition graph  $ICG(G, \pi)$  of  $\pi = \{V_1, \dots, V_k\}$  (and of  $G$ ) has the vertex set  $\{V_1, \dots, V_k\}$ , vertices  $V_i$  and  $V_j$  being adjacent if  $V_i$  and  $V_j$  form an independent coalition in  $G$ . In this paper, a large family of graphs with  $IC(G) = 0$  is described and graphs  $G$

with  $IC(G) \in \{n(G), n(G) - 1\}$  characterized. Some properties of  $ICG(G, \pi)$  are presented. The independent coalition graphs of paths are characterized, and the independent coalition graphs of cycles described.

**Keywords:** dominating set; independent set; independent coalition; independent coalition number; independent coalition graph

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## 1 Introduction

Let  $G = (V(G), E(G))$  be a graph. A *coalition* in  $G$  consists of two disjoint sets  $V_1$  and  $V_2$  of vertices, such that neither  $V_1$  nor  $V_2$  is a dominating set, but the union  $V_1 \cup V_2$  is a dominating set of  $G$ . A *coalition partition* of  $G$ , *c-partition* of  $G$  for short, is a partition  $\{V_1, \dots, V_k\}$  of  $V(G)$  such that for every  $i \in [k]$ , either the set  $V_i$  consists of a single dominating vertex of  $G$ , or  $V_i$  forms a coalition with some other part  $V_j$ . Coalition partitions were introduced in 2020 in [8] and already extensively researched in [3, 4, 9–12]. Very recently, total coalition partitions of graphs have started to be explored in [1, 5] and connected coalition partitions of graphs in [2].

It is a generally accepted fact that the central concepts of graph domination are the domination itself, the total domination, and the connected domination, see the very comprehensive 2023 book [13] on the core concepts in domination which focuses precisely on these three topics. It therefore makes sense also to explore independent c-partitions in graphs which are defined just as c-partitions, except that in addition independence is required of the sets involved. More precisely, an *independent coalition* in  $G$  consists of two disjoint, independent vertex sets  $V_1$  and  $V_2$ , such that neither  $V_1$  nor  $V_2$  is a dominating set, but the union  $V_1 \cup V_2$  is an independent dominating set of  $G$ . An *independent c-partition* of  $G$  is a partition  $\{V_1, \dots, V_k\}$  of  $V(G)$  such that for every  $i \in [k]$ , either the set  $V_i$  consists of a single dominating vertex of  $G$ , or  $V_i$  forms an independent coalition with some other part  $V_j$ . Independent coalitions were introduced/mentioned in [8], see also [11]. However, this concept was first explored in more detail in [15]. In this paper we continue the research in this direction and proceed as follows.

In the rest of this section we define concepts needed and introduce the relevant notation. In Section 2, we consider graphs with extremal independent coalition numbers. We first consider graphs  $G$  with  $IC(G) = 0$  and conclude the section by graphs  $G$  with  $IC(G) \in \{n(G), n(G) - 1\}$ . We define and study the independent coalition graph of a graph in Section 3. In Section 4 we characterize the independent coalition graphs of paths, while in Section 5 we describe the independent coalition graphs of cycles.

Let  $G$  be a graph and  $S \subseteq V(G)$ . Then  $S$  is a *dominating set* if every vertex in  $V(G) \setminus S$  has a neighbor in  $S$ , and  $S$  is an *independent set* if no two vertices

from  $S$  are adjacent. By an *independent dominating set* we mean a set that is both, dominating and independent. A vertex  $v$  of  $G$  which is adjacent to every other vertex, is a *dominating vertex* of  $G$ . The *independent coalition number*,  $IC(G)$ , of a graph  $G$  is the maximum order an independent  $c$ -partition in  $G$ . If a graph  $G$  does not admit an independent  $c$ -partition, then we set  $IC(G) = 0$ .

An *idomatic partition* of a graph is a partition of the vertices into independent dominating sets. Such partitions seems to be considered for the first time in 2000 in the paper [7] under the name *fall colorings*. Indeed, an idomatic partition is a proper coloring such that every vertex has every color in its open neighborhood. We say that a graph is *idomatic*, if its vertex set can be partitioned into independent dominating sets.

If  $G$  and  $H$  are graphs and  $k$  a positive integer, then  $G \cup H$  denotes the disjoint union of  $G$  and  $H$  and  $kG$  the disjoint union of  $k$  copies of  $G$ . In addition,  $G + H$  is the join of  $G$  and  $G$ , that is, the graph obtained from  $G \cup H$  by adding all edges  $gh$ , where  $g \in V(G)$  and  $h \in V(H)$ . Finally, the order of a graph  $G$  will be denoted by  $(G)$ , and  $[n]$  stands for the set  $\{1, \dots, n\}$ .

## 2 Graphs $G$ with $IC(G) \in \{0, n(G), n(G) - 1\}$

In the seminal paper [8] a problem was posed whether every graph  $G$  admits an independent  $c$ -partition, that is, whether for every graph  $G$  we have  $IC(G) > 0$ . In [15], Samadzadeh and Mojdeh answered this question in negative by demonstrating that there exist graphs  $G$  with  $IC(G) = 0$  as follows. Let  $X_n$ ,  $n \geq 4$ , be the graph obtained from  $K_n$  with  $V(K_n) = \{v_1, \dots, v_n\}$  and two additional vertices  $v_{n+1}, v_{n+2}$  by adding the edges  $v_n v_{n+1}$ ,  $v_n v_{n+2}$ , and  $v_{n-1} v_{n+1}$ . See Fig. 1, where  $X_5$  is drawn. Then it was proved in [15] that for  $n \geq 4$ ,  $IC(X_n) = 0$ .

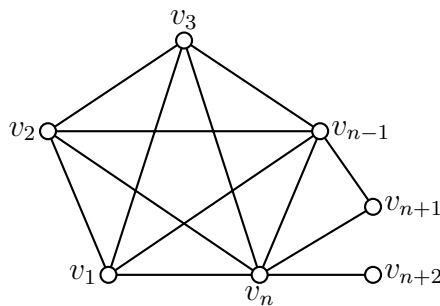


Figure 1: The graph  $X_5$ ; so  $n = 5$  in the figure

In this section we present a significantly larger family of graphs which do not admit independent  $c$ -partitions. For it, the following fact that immediately follows

by the definition of an independent c-partition will be useful.

**Lemma 2.1** *Let  $x$  be a dominating vertex of a graph  $G$ . Then  $IC(G) > 0$  if and only if  $IC(G - x) > 0$ .*

For a graph  $G$ , the graph  $\widehat{G}$  is a graph defined as follows. Its vertex set is  $V(\widehat{G}) = V(G) \cup \{x, y\}$  and the edge set is  $E(\widehat{G}) = E(G) \cup \{xu : u \in V(G)\} \cup \{xy\}$ . Now the main result of this section reads as follows.

**Theorem 2.2** *Let  $G$  be a graph. Then  $IC(\widehat{G}) > 0$  if and only if  $G = H \cup sK_1$  for some  $s \geq 0$  and some idomatic graph  $H$ .*

**Proof.** We use the notation from the definition of  $\widehat{G}$ , hence  $V(\widehat{G}) = V(G) \cup \{x, y\}$ , where  $x$  is a dominating vertex of  $\widehat{G}$ .

Assume first that  $IC(\widehat{G}) > 0$ . As  $x$  is a dominating vertex of  $\widehat{G}$ , Lemma 2.1 implies that  $IC(G \cup K_1) > 0$ . Set  $G' = G \cup K_1$ , where we may assume that  $V(K_1) = \{y\}$ . Let  $\pi = \{V_1, \dots, V_k\}$  be an independent c-partition of  $V(G')$  and assume without loss of generality that  $y \in V_1$ . We distinguish two cases.

**Case 1:**  $V_1 = \{y\}$ .

Since  $V_j, j \in \{2, \dots, k\}$ , must dominate  $G'$  together with some other set from  $\pi$ , the latter set must necessarily be  $V_1$ . Since  $|V_1| = 1$  this in turn implies that  $V_j$  dominates  $G$ . It follows that  $\{V_2, \dots, V_k\}$  is an idomatic partition of  $G$  and consequently  $G$  is an idomatic graph. Clearly, we can write it as  $G = G \cup 0K_1$ .

**Case 2:**  $|V_1| \geq 2$ .

In this case  $V_1 \cap V(G) \neq \emptyset$ , which implies that  $V_1 \cap V(G)$  is an independent set of  $G$  but it is not a dominating set of  $G$ , for otherwise  $V_1$  would dominate  $G'$ . Moreover,  $V_1 \cup V_j$  must be an independent dominating set of  $G'$  for every  $j \in \{2, \dots, k\}$ . This in particular implies that each vertex of  $V_1 \cap V(G)$  must be an isolated vertex of  $G$  and that each  $V_j, j \geq 2$ , is an independent dominating set of  $H = G[V(G) \setminus V_1]$ . If  $|V_1 \cap V(G)| = s$ , then we can conclude that  $G = H \cup sK_1$ , where  $H$  is an idomatic graph.

Conversely, assume that  $G = H \cup sK_1$ , where  $H$  is an idomatic graph and  $s \geq 0$ . Let  $\{V_1, \dots, V_k\}$  be an idomatic partition of  $H$  and let  $S$  be the set of isolated vertices of  $G$ , where  $S = \emptyset$  in case  $s = 0$ . Then we claim that  $\{S \cup \{y\}, V_1, \dots, V_k\}$  is an independent c-partition of  $G'$ . Indeed,  $S \cup \{y\}$  is clearly an independent set of  $G'$  and as each  $V_i, i \geq 1$ , is a independent dominating set of  $H$ , we also have that  $(S \cup \{y\}) \cup V_i$  is an independent dominating set of  $G'$ . By Lemma 2.1 we conclude that  $IC(\widehat{G}) > 0$ .  $\square$

Let  $H_n, n \geq 4$ , be the graph obtained from  $K_{n-1}$  by attaching a pendant vertex to one of the vertices of  $K_{n-1}$ . Then  $\widehat{H}_n \cong X_n$ . Since the graph  $H_n$  is clearly not idomatic, Theorem 2.2 implies that  $IC(X_n) = 0$ .

Moreover, Theorem 2.2 yields a large variety of graphs which admit no independent coalition partition. If  $G$  is an arbitrary connected and not idomatic graph, then the theorem implies that  $IC(\widehat{G}) = 0$ . For instance, a cycle  $C_n$  is not idomatic if and only if  $n$  is odd and not congruent modulo 3, cf. [14]. Additional families of graphs that are not idomatic were constructed in [6], for instance graphs  $G$  with  $\chi(G) > \delta(G) + 1$ . For more information on the variety of domination partitions see [13, Chapter 12].

We now turn our attention to graphs with (almost) largest possible extremal independent coalition number, that is, to graphs  $G$  with  $IC(G) \in \{n(G), n(G) - 1\}$ .

The following observation follows directly from definitions, but it is useful because it enables a direct, polynomial verification whether  $IC(G) = n(G)$  holds for a given graph  $G$ .

**Observation 2.3** *Let  $G$  be a graph without dominating vertices. Then  $IC(G) = n(G)$  if and only if for any  $v \in V(G)$ , there exists  $z \in V(G)$  with  $z \notin N[v]$  such that  $V(G) = N[v] \cup N[z]$ .*

If  $G$  has  $k$  dominating vertices,  $IC(G) = k + IC(G')$ , where  $G'$  is obtained from  $G$  by removing the  $k$  dominating vertices. Note that  $G'$  has no dominating vertices and hence Observation 2.3 can be applied to  $G$ . It is then straightforward to see that checking whether  $IC(G) = n(G)$  holds for a graph  $G$  can be performed in  $\mathcal{O}(n(G)^3)$  time.

To check whether  $IC(G) = n(G) - 1$  holds, we can use the following proposition. Its proof is straightforward and hence not included.

**Proposition 2.4** *Let  $G$  be a graph without dominating vertices. Then  $IC(G) = n(G) - 1$  if and only if the following properties hold.*

- (i) *There exists a vertex  $a \in V(G)$  such that for all vertices  $b \neq a$  the set  $\{a, b\}$  is not an independent dominating set.*
- (ii) *There exists two nonadjacent vertices  $x, y \in V(G)$ , such that there exists a vertex  $w \in V(G)$ ,  $w \neq x, y$ , such that  $\{w, x, y\}$  is an independent dominating set and for every vertex  $u \in V(G)$  with  $u \neq x, y$ , the set  $\{u, x, y\}$  is an independent dominating set or there exists a vertex  $v \neq u$  such that  $\{u, v\}$  is an independent dominating set.*

Note that the condition (i) of Proposition 2.4 rules out the possibility  $IC(G) = n(G)$ , and then (ii) checks whether  $IC(G) = n(G) - 1$  holds. As we already mentioned, checking whether  $IC(G) = n(G)$  holds (equivalently, condition (i)) can be done in  $\mathcal{O}(n(G)^3)$  time. As for condition (ii), its testing for each pair of vertices  $x$  and  $y$  can be done in  $\mathcal{O}(n(G)^3)$  time, leading to an  $\mathcal{O}(n(G)^5)$  time algorithm for checking whether  $IC(G) = n(G) - 1$  holds.

### 3 Independent coalition graphs

Given an independent  $c$ -partition  $\pi = \{V_1, \dots, V_k\}$  of a graph  $G$ , we can associate to it a natural derived graph as follows. The *independent coalition graph*  $\text{ICG}(G, \pi)$  of  $\pi$  (and of  $G$ ) has the vertex set  $V(\text{ICG}(G, \pi)) = \{V_1, \dots, V_k\}$ , and vertices  $V_i$  and  $V_j$  are adjacent if  $V_i$  and  $V_j$  form an independent coalition in  $G$ . In the following we present some general properties of the independent coalition graph.

**Proposition 3.1** *Let  $G$  be a graph and  $\pi$  be an independent  $c$ -partition of  $G$  with  $|\pi| = k$ . Then the following holds.*

- (i)  $\Delta(\text{ICG}(G, \pi)) \leq \Delta(G) + 1$ .
- (ii) *If  $k \geq \delta(G) + 2$ , then  $\alpha(\text{ICG}(G, \pi)) \geq k - \delta(G) - 1$ .*

**Proof.** Let  $\pi = \{V_1, \dots, V_k\}$ .

(i) There is nothing to prove when  $\Delta(\text{ICG}(G, \pi)) = 0$ . Hence assume that  $\Delta(\text{ICG}(G, \pi)) > 0$  and let  $V_i$  be a vertex of  $\text{ICG}(G, \pi)$  with  $\deg_{\text{ICG}(G, \pi)}(V_i) = \Delta(\text{ICG}(G, \pi)) \geq 1$ . Then  $|V_i| > 1$  and  $V_i$  is not a dominating set. Therefore, there exists a vertex  $v \in V(G)$  with no neighbor in  $V_i$ . If  $V_i$  and  $V_j$  form an independent coalition, then  $V_i \cup V_j$  is an independent dominating set, and therefore  $v$  has at least one neighbor in  $V_j$ . It follows that

$$\Delta(\text{ICG}(G, \pi)) = \deg_{\text{ICG}(G, \pi)}(V_i) \leq \deg_G(v) + 1 \leq \Delta(G) + 1,$$

which proves (i).

(ii) Assume now that  $k \geq \delta(G) + 2$  and let  $v$  be a vertex of  $G$  with  $\deg_G(v) = \delta(G)$ . As the sets from  $\pi$  are independent, at most  $\delta(G) + 1$  of them contain a vertex from  $N_G[v]$ . Consequently, at least  $k - \delta(G) - 1 \geq 1$  sets of  $\pi$  do not contain a vertex from  $N_G[v]$ . It follows that no two of these  $k - \delta(G) - 1$  sets form an independent coalition and hence as the vertices of  $\text{ICG}(G, \pi)$  form an independent set. We conclude that  $\alpha(\text{ICG}(G, \pi)) \geq k - \delta(G) - 1$ .  $\square$

Consider the graph  $G$  obtained from the complete graph  $K_n$  with a pendant edge. This graph  $G$  satisfies the equality in Proposition 3.1(i). Also the path graph  $P_n$  satisfies the equality of Proposition 3.1(ii).

**Proposition 3.2** *Let  $G$  be a graph with  $\delta(G) = 1$  and let  $\pi$  be an independent  $c$ -partition of  $G$  with  $|\pi| = k \geq 3$ . Then  $\text{ICG}(G, \pi)$  is a spanning subgraph of  $K_{2, k-2}$ .*

**Proof.** Let  $\pi = \{V_1, \dots, V_k\}$ , let  $x$  be a vertex of  $G$  of degree 1, and let  $y$  be its only neighbor. We may without loss of generality assume that  $x \in V_1$  and  $y \in V_2$ . Then  $\{V_1, V_2\} \notin E(\text{ICG}(G, \pi))$ . In addition, if  $i, j \geq 3$ ,  $i \neq j$ , then  $x \notin N_G[V_i \cup V_j]$  which in turn implies that  $\{V_i, V_j\} \notin E(\text{ICG}(G, \pi))$ . Hence  $\{V_3, \dots, V_k\}$  is an independent set of  $\text{ICG}(G, \pi)$  which together with the fact that  $\{V_1, V_2\} \notin E(\text{ICG}(G, \pi))$  implies the result.  $\square$

## 4 Independent coalition graphs of paths

In this section we consider independent coalitions in paths. Their independent coalition numbers have already been determined as follows.

**Theorem 4.1** [15] *If  $n \geq 1$ , then*

$$IC(P_n) = \begin{cases} n; & n \leq 4, \\ 4; & n = 5, \\ 5; & n \in \{6, 7, 8, 9\}, \\ 6; & n \geq 10. \end{cases}$$

We say that a graph  $G$  is an ICG-graph if  $G$  is isomorphic to the independent coalition graphs of some graph. In this section we complement Theorem 4.1 by determining the ICG-graphs of paths.

**Theorem 4.2** *A graph  $G$  is an ICG-graph of some path if and only if*

$$G \in \{P_1, P_4, P_5, 2P_1, 2P_2, 2P_3, P_1 \cup P_2, P_2 \cup P_3\}.$$

**Proof.** Throughout the proof we will assume that  $x_1, \dots, x_n$  are consecutive vertices of  $P_n$ ,  $n \geq 1$ . Further, we will represent an independent c-partition  $\pi = \{V_1, \dots, V_k\}$  of  $P_n$  by the vector  $f(\pi) = (f_1(\pi), \dots, f_n(\pi))$ , where  $x_i \in V_{f_i(\pi)}$ . As an example consider the independent c-partition  $\pi = \{\{x_1, x_5\}, \{x_2, x_4\}, \{x_3\}, \{x_6\}\}$  of  $P_6$ . Then  $\pi$  is represented by the vector  $f(\pi) = (1, 2, 3, 2, 1, 4)$ , where, for instance,  $f_4(\pi) = 2$  means that  $x_4 \in V_2$ .

We first demonstrate that each of the graphs listed in the statement of the theorem is an ICG-graph of some path. Considering  $P_1$  and  $P_2$  we obtain  $P_1$  and  $2P_1$  as ICG-graphs. For the remaining six graphs, here are instances of their realizations:

- $P_4$ : the path  $P_7$  with the independent c-partition  $(1, 2, 3, 4, 3, 2, 1)$ ;
- $P_5$ : the path  $P_{11}$  with the independent c-partition  $(1, 2, 1, 5, 4, 3, 5, 4, 3, 2, 1)$ ;
- $2P_2$ : the path  $P_6$  with the independent c-partition  $(1, 2, 3, 2, 1, 4)$ ;
- $2P_3$ : the path  $P_{10}$  with the independent c-partition  $(2, 1, 5, 6, 1, 2, 4, 3, 2, 1)$ ;
- $P_1 \cup P_2$ : the path  $P_3$  with the independent c-partition  $(1, 2, 3)$ ;
- $P_2 \cup P_3$ : the path  $P_9$  with the independent c-partition  $(1, 2, 3, 4, 2, 1, 5, 1, 2)$ .

It remains to prove that no other graph but the above graphs is an ICG-graph of some path. Since we have settled above all the cases for  $P_n$ ,  $n \leq 3$ , we may assume in the rest that  $n \geq 4$ .

We first recall that in [8, Lemma 1], it has been proved that  $C(P_n) \leq 6$ . Since  $IC(P_n) \leq C(P_n)$ , we have  $IC(P_n) \leq 6$ . Consequently, Proposition 3.2 implies that independent coalition graphs of paths are spanning subgraphs of  $K_{2,r}$  with  $r \leq 4$ . In addition, since we have assumed that  $n \geq 4$ , no independent coalition graph of  $P_n$  contains isolated vertices. By inspection we find the following 20 non-isomorphic spanning subgraphs of  $K_{2,r}$ , where  $r \leq 4$  with minimum degree at least 1:

$$P_1, P_4, P_5, 2P_1, 2P_2, 2P_3, P_1 \cup P_2, P_2 \cup P_3, \quad (1)$$

$$P_3, K_{2,2}, K_{2,3}, K_{2,4}, K_{2,3} - e, K_{2,4} - e, K_{1,3} \cup P_2, F_1, F_2, F_3, F_4, F_5, \quad (2)$$

where the graphs  $F_i$ ,  $i \in [5]$ , are depicted in Fig. 2.

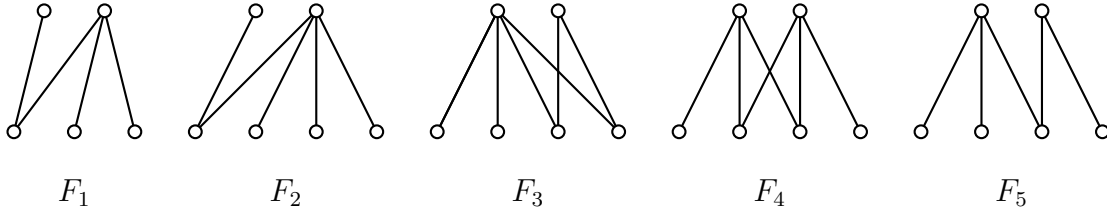


Figure 2: The graphs  $F_i$ ,  $i \in [5]$

For the graphs from (1) we have established above that they are ICG-graphs of paths, hence we need to prove that neither of the graphs from (2) is an ICG-graph of some path.

Consider first  $P_3$  and suppose that it is a an ICG-graph of some path  $P_n$  with an independent  $c$ -partition  $\pi = \{V_1, V_2, V_3\}$ . We may assume without loss of generality that  $f_1(\pi) = 1$  and  $f_2(\pi) = 2$ . Then  $V_2 \cup V_3$  is an independent dominating set, hence  $f_3(\pi) = 1$ . Since also  $V_1 \cup V_3$  is an independent dominating set, we have  $f_4(\pi) = 2$ . Continuing in this manner we conclude that  $V_3 = \emptyset$ , a contradiction.

Consider second  $K_{2,3}$  and suppose that it is a an ICG-graph of some path  $P_n$  with an independent  $c$ -partition  $\pi = \{V_1, \dots, V_5\}$ . Assume without loss of generality that  $\{1, 2\}, \{3, 4, 5\}$  is the bipartition of  $K_{2,3}$ , where  $i \in V_i$  for  $i \in [5]$ . Let  $j$  be an arbitrary index such that  $f_j(\pi) = 1$ . Then  $f_{j-1}(\pi) = 2$  (if  $j-1 \geq 1$ ) and  $f_{j+1}(\pi) = 2$  (if  $j+1 \leq n$ ). In this way we see that  $V_3 = V_4 = V_5 = \emptyset$ , a contradiction. The same argument applies to  $K_{2,2}$  and  $K_{2,4}$ .

Consider next  $K_{2,3} - e$  and suppose that it is a an ICG-graph of some path  $P_n$  with an independent  $c$ -partition  $\pi = \{V_1, \dots, V_5\}$ . Assume without loss of generality that  $\{1, 2\}, \{3, 4, 5\}$  is the bipartition of  $K_{2,3}$ , where  $\{2, 5\} \notin E(K_{2,3} - e)$ , and  $i \in V_i$



for  $i \in [5]$ . Let  $j$  be such that  $f_j(\pi) = 1$ . Then  $f_{j-1}(\pi) = 2$  (if  $j - 1 \geq 1$ ) and  $f_{j+1}(\pi) = 2$  (if  $j + 1 \leq n$ ). Assume without loss of generality that  $j + 1 < n - 1$  and that  $\ell \geq j + 2$  is the smallest index such that  $f_\ell(\pi) \in \{3, 4, 5\}$ . Then necessary  $f_\ell(\pi) = 3$ . Thus we have  $f_{\ell-2}(\pi) = 1$ ,  $f_{\ell-1}(\pi) = 2$ , and  $f_\ell(\pi) = 3$ , which in turn implies that  $f_{\ell+1}(\pi) \in \{4, 5\}$  (if  $\ell + 1 \leq n$ ). But then at least one of  $V_1 \cup V_4$  and  $V_1 \cup V_5$  is not a dominating set because one of these sets does not dominate  $x_\ell$ . This contradiction proves that  $K_{2,3} - e$  is not an ICG-graph of some path. A parallel argument can be used also for  $K_{2,4} - e$  as well as for  $F_1$ ,  $F_2$ , and  $F_3$ . (Note that in each of these graphs there exists a vertex from the smaller bipartition set adjacent to all the vertices from the other bipartition set.)

It remains to consider the graphs  $K_{1,3} \cup P_2$ ,  $F_4$ , and  $F_5$ . As the arguments are similar, let us consider in detail only  $K_{1,3} \cup P_2$ . Suppose on the contrary that  $K_{1,3} \cup P_2$  is an ICG-graph of some path  $P_n$  with an independent  $c$ -partition  $\pi = \{V_1, \dots, V_6\}$ . Assume without loss of generality that the vertices of  $K_{1,3}$  are from [4] with 1 being the vertex of degree 3, and that  $V(P_2) = \{5, 6\}$ , where  $i \in V_i$  for  $i \in [6]$ . Consider a vertex  $x_j$  with  $f_j(\pi) = 1$  and assume without loss of generality that  $j < n - 1$ . Then  $f_{j+1}(\pi) \in \{5, 6\}$ . If  $f_{j+2}(\pi) = 1$ , we repeat the pattern. Hence assume that  $f_{j+2}(\pi) \in \{2, 3, 4\}$ . But now no matter what is the value of  $f_{j+3}(\pi)$  (if  $j + 3 \leq n$ ), at least one of the sets  $V_1 \cup V_2$ ,  $V_1 \cup V_3$ , and  $V_1 \cup V_4$  is not a dominating set because at least one of them does not dominate  $x_{j+2}$ . This contradiction completes the argument for  $K_{1,3} \cup P_2$ .  $\square$

## 5 Independent coalition graphs of cycles

Here we describe the independent coalition graphs of cycles, henceforth called ICG-graphs of cycles. Similar to the result for paths, the number of ICG-graphs of cycles is finite. This fact follows from the following known result that gives the independent coalition numbers of cycles.

**Theorem 5.1** [15, Theorem 3.11] *If  $n \geq 3$ , then*

$$IC(C_n) = \begin{cases} n; & n \leq 6, \\ 5; & n = 7, \\ 6; & n \geq 8. \end{cases}$$

Just as done for paths, we denote the consecutive vertices of  $C_n$  by  $x_1, \dots, x_n$  and represent an independent  $c$ -partition  $\pi = \{V_1, \dots, V_k\}$  of  $C_n$  by the vector  $f(\pi) = (f_1(\pi), \dots, f_n(\pi))$ , where  $x_i \in V_{f_i(\pi)}$ . As an example consider the independent  $c$ -partition  $\pi = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_5\}, \{x_6\}, \{x_7\}\}$  of  $C_7$ . Then  $\pi$  is represented by the vector  $f(\pi) = (1, 2, 1, 2, 3, 4, 5)$ , where, for instance,  $f_5(\pi) = 3$  means that  $x_5 \in V_3$ .

Clearly, the only cycle whose ICG-graph contains an isolated vertex is  $C_3$ , more precisely,  $\text{ICG}(C_3, \pi) = 3K_1$ , where  $\pi$  is the unique independent  $c$ -partition of  $C_3$ . For longer cycles we have the following:

**Proposition 5.2** *Let  $\pi$  be an independent  $c$ -partition of  $C_n$ ,  $n \geq 4$ . Then  $\text{ICG}(C_n, \pi)$  is a spanning subgraph of one of the graphs  $(K_1 \cup K_2) + K_1$ ,  $(K_1 \cup K_2) + 2K_1$ , and  $(K_1 \cup K_2) + 3K_1$ .*

**Proof.** Let  $\pi = \{V_1, \dots, V_k\}$  be an independent  $c$ -partition of  $C_n$ . Since  $n \geq 4$ ,  $\text{ICG}(C_n)$  has no isolated vertices. Moreover,  $k = |\pi| \geq 4$ . Indeed, if we would have  $|\pi| = 3$ , then there exist three consecutive vertices of  $C_n$  such that they respectively belong to the three parts of  $\pi$ . We may assume without loss of generality that  $f_i(\pi) = i$  for  $i \in [3]$ . But then neither  $V_2 \cup V_1$  nor  $V_2 \cup V_3$  is an independent set, a contradiction.

We have thus seen that  $k \geq 4$ . On the other hand,  $k \leq 6$  Theorem 5.1. Just as above, there exist three consecutive vertices of  $C_n$  such that they respectively belong to the three parts of  $\pi$  and we may assume that  $f_i(\pi) = i$  for  $i \in [3]$ . Then  $V_2V_1 \notin E(\text{ICG}(C_n, \pi))$  and  $V_2V_3 \notin E(\text{ICG}(C_n, \pi))$ . It is possible however that  $V_1V_3 \in E(\text{ICG}(C_n, \pi))$ . If  $k = 4$ , then  $\text{ICG}(C_n, \pi)$  is a spanning subgraph of  $(K_1 \cup K_2) + K_1$ . Assume  $k = 5$ . Then  $V_4 \cup V_5$  is not a dominating set since the union does not dominate  $x_2$ . In this case  $\text{ICG}(C_n, \pi)$  is a spanning subgraph of  $(K_1 \cup K_2) + 2K_1$ . Assume finally that  $k = 6$ . Then by the same argument  $V_j \cup V_{j'}$  is not a dominating set for any  $j, j' \in \{4, 5, 6\}$ ,  $j \neq j'$ . Hence in this case  $\text{ICG}(C_n, \pi)$  is a spanning subgraph of  $(K_1 \cup K_2) + 3K_1$ .  $\square$

The variety of the ICG-graphs of cycles thus appears larger than the ICG-graphs of paths. Therefore, we will not make a precise analysis of which graphs from Proposition 5.2 are ICG-graphs of cycles. Instead, we conclude with two ICG-graphs of cycles which are not ICG-graphs of paths.

- $C_5$ : the cycle  $C_5$  with the independent  $c$ -partition  $(1, 2, 3, 4, 5)$ ;
- $3P_2$ : the cycle  $C_6$  with the independent  $c$ -partition  $(1, 2, 3, 4, 5, 6)$ .

Other realizations of the same graph can also exist. For instance, the graph  $3P_2$  can be realized as the ICG-graph of the cycle  $C_9$  with the independent  $c$ -partition  $(1, 3, 5, 1, 3, 5, 2, 4, 6)$  and as the ICG-graph of the cycle  $C_{12}$  with the independent  $c$ -partition  $(1, 3, 5, 1, 3, 5, 2, 4, 6, 2, 4, 6)$ .

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## References

- [1] S. Alikhani, D. Bakhshesh, H. Golmohammadi, Total coalitions in graphs, [arXiv:2211.11590v2 \[math.CO\]](https://arxiv.org/abs/2211.11590v2) (10 Dec 2022).
- [2] S. Alikhani, D. Bakhshesh, H. Golmohammadi, E.V. Konstantinova, Connected coalitions in graphs, *Discuss. Math. Graph Theory* (2023) [doi.org/10.7151/dmgt.2509](https://doi.org/10.7151/dmgt.2509).
- [3] S. Alikhani, H. Golmohammadi, E. Konstantinova, Coalition of cubic graphs of order at most 10, *Commun. Comb. Optim.* (2023) [10.22049/cco.2023.28328.1507](https://doi.org/10.22049/cco.2023.28328.1507).
- [4] D. Bakhshesh, M.A. Henning, D. Pradhan, On the coalition number of trees, *Bull. Malays. Math. Sci. Soc.* 46 (2023) Paper No. 95.
- [5] J. Barát, Z.L. Blázsik, General sharp upper bounds on the total coalition number, *Discuss. Math. Graph Theory* (2023) [doi.org/10.7151/dmgt.2511](https://doi.org/10.7151/dmgt.2511).
- [6] E.J. Cockayne, S.T. Hedetniemi, Disjoint independent dominating sets in graphs, *Discrete Math.* 15 (1976) 213–222.
- [7] J.E. Dunbar, S.M. Hedetniemi, S.T. Hedetniemi, D.P. Jacobs, J. Knisely, R.C. Laskar, D.F. Rall, Fall colorings of graphs, *J. Combin. Math. Combin. Comput.* 33 (2000) 257–273.
- [8] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Introduction to coalitions in graphs, *AKCE Int. J. Graphs Combin.* 17 (2020) 653–659.
- [9] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Upper bounds on the coalition number, *Australas. J. Combin.* 80 (2021) 442–453.
- [10] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Self-coalition graphs, *Opuscula Math.* 43 (2023) 173–183.

- [11] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Coalition graphs, *Commun. Combin. Optim.* 8 (2023) 423–430.
- [12] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Coalition graphs of paths, cycles, and trees, *Discuss. Math. Graph Theory* 43 (2023) 931–946.
- [13] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, *Domination in Graphs: Core Concepts*, Springer, 2023.
- [14] H. Kaul, C. Mitillos, On graph fall-coloring: Existence and constructions, *Graphs Combin.* 35 (2019) 1633–1646.
- [15] M.R. Samadzadeh, A.M. Doost, Independent coalition in graphs, [arXiv:2306.02079](https://arxiv.org/abs/2306.02079) [math.CO] (3 Jun 2023).